INTEGRAL EQUATIONS FOR THE TEMPERATURE DISTRIBUTION IN THE PLANAR FLOW OF A NON-NEWTONIAN MEDIUM

V. I. Naidenov

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Comparatively little is known analytically on problems of convective heat exchange in a viscoplastic medium such as suspensions, solutions, and polymer melts with rheological properties dependent on temperature. Because of the complexity of the equations of motion, numerical methods have prevailed. Using the self-modeling equations of [2] we obtain nonlinear integral equations describing the temperature distribution in a non-Newtonian medium for the case of planar flow.

We consider a fluid moving between two planes separated by distance 2h $(-h \leqslant y \leqslant h)$. Let the temperature of the walls of the channel vary linearly with the longitudinal coordinate x, $T_w = T_0 + Ax$.

As shown in [2], in certain cases uniform motion occurs, despite the dependence of the viscosity on the coordinates x and y. Therefore we consider the system of equations

$$\partial p/\partial x = \partial\tau/\partial y,\ \partial p/\partial y = \partial\tau/\partial x,\ \lambda\Delta T = \rho c_p \partial T/\partial t + \rho c_p u \partial T/\partial x. \tag{1}$$

We use the rheological equation

$$\tau = k_0 \left|\frac{du}{dy}\right|^{n-1} e^{-\gamma(T-T_0)} \frac{du}{dy}, \tag{2}$$

where k_0 is a constant determined by the consistency of the medium, n is the flow index (n > 1 corresponds to a dilating fluid, n < 1 to a pseudoplastic fluid, and u(y) is the fluid velocity. We assume that the temperature of the fluid can be written as the sum of two terms

$$T(x,\ y) = T_w(x) + T_1(y). \tag{3}$$

Substituting (2) and (3) into the stationary system of equations (1) and performing straightforward calculations, we get the following boundary value problem

$$\theta''' = -\sqrt[n]{\frac{D\,\mathrm{sh}\,\delta\xi}{\mathrm{sh}\,\delta}}\, e^{\frac{H}{n}\theta},$$
$$\theta(1) = 0,\ \theta'(1) = 1,\ \theta''(1) = 0,\ \theta'(0) = 0, \tag{4}$$

where

$$\xi = \frac{y}{h};\ \theta = \frac{T_1\lambda}{q_w h};\ H = \frac{\gamma q_w h}{\lambda};\ \mathrm{Pe} = \frac{u_0 h \rho c_p}{\lambda};\ \delta = \frac{H}{\mathrm{Pe}};$$
$$D = 0{,}5\,\mathrm{Re}_w\lambda_w;\ \mathrm{Re}_w = \frac{u_0^{2-n}\rho h^n}{k_w};\ \lambda_w = \frac{|\tau_w|}{0{,}5\rho u_0^2};$$

Here q_w is the heat flux at the walls, τ_w is the frictional stress on the walls, u_0 is the average flow velocity, k_w is the consistency constant of the medium evaluated at the wall temperature, λ_w is the local drag coefficient, and Pe, Re_w are the Peclet and Reynolds numbers.

For a given average flow velocity u_0, the constant D will be unknown and along with the three constants of integration of the nonlinear differential equation (4) will be determined by the four boundary conditions. After the solution $\theta(\xi)$ of (4) is found, the fluid velocity can be obtained from the relation $u = u_0 w(\xi) = u_0\theta''(\xi)$, and the Nusselt number, calculated with respect to the weighted mean temperature of the fluid, is given by

$$\frac{1}{\mathrm{Nu}} = \frac{1}{2}\int_0^1 (\theta'(\xi))^2 d\xi.$$

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Equation (4) can be solved efficiently by an approximate series expansion, as discussed below. We consider the case of large Pe; this is relevant because we consider the medium to be weakly heat conducting and therefore large Peclet numbers can be reached even for small flow velocities. In this case $\delta \to 0$ and the boundary value problem (4) reduces to

$$\theta''' = -D^m \xi^m e^{Hm\theta},\ m = 1/n, \qquad (5)$$
$$\theta(1) = 0,\ \theta'(1) = 1,\ \theta''(1) = 0,\ \theta'(0) = 0.$$

Differentiating (5) with respect to ξ and letting $q(\xi) = \theta'(\xi)$, we obtain a boundary value problem for the heat flux

$$q''' = (m/\xi)q'' + Hmq''q,\ q(1) = 1,\ q'(1) = 0,\ q(0) = 0. \qquad (6)$$

We look for the solution of (6) in the form of a convergent [3] series in powers of the small parameter Hm:

$$q(\xi) = \sum_{k=0}^{\infty} q_k(\xi)(Hm)^k, \qquad (7)$$
$$q_0(1) = 1,\ q_0'(1) = 0,\ q_0(0) = 0;\ q_i(1) = q_i'(1) = q_i(0) = 0,\ i > 0.$$

Substituting (7) into (6) and equating coefficients of identical powers of Hm, we obtain a system of linear differential equations which can be solved exactly, The first two terms of series (7) are

$$q_0(\xi) = [(m+2)\xi - \xi^{m+2}]/(m+1),$$
$$q_1(\xi) = \frac{\xi^{2m+5}}{2(m+1)(2m+5)(m+3)} - \frac{(m+2)^2\xi^{m+4}}{2(m+1)(m+3)(m+4)} +$$
$$+ \frac{(m+2)(4m^3+30m^2+70m+44)}{4(m+1)^2(2m+5)(m+3)(m+4)}\xi^{m+2} - \frac{(4m^3+25m^2+49m+28)\xi}{2(m+1)^2(2m+5)(m+3)(m+4)}.$$

From this solution we can determine some of the dynamical and thermal characteristics of the flow:

(a) drag coefficient

$$\frac{1}{2}\mathrm{Re}_w\lambda_w = \left[m + 2 + \frac{Hm(m+2)(8m^3+56m^2+120m+72)}{4(m+1)(2m+5)(m+3)(m+4)}\right]^{1/m};$$

(b) fluid velocity along the axis of the channel

$$w(0) = \frac{m+2}{m+1} - \frac{Hm(4m^3+2m^2+49m+28)}{2(m+1)^2(2m+5)(m+3)(m+4)};$$

(c) Nusselt number for a Newtonian fluid (m = 1)

$$\frac{1}{\mathrm{Nu}} = \frac{17}{70} - \frac{377}{42425}H;$$

(d) Nusselt number for an ideal dilating medium (small values of m)

$$\frac{1}{\mathrm{Nu}} = \frac{8}{30} - \frac{343}{25\,200}Hm.$$

The above solution will be correct for small values of H, which is proportional to the thermal load of the walls. When H is large the boundary value problem (4) can be integrated numerically. The results of these calculations are shown in Figs. 1-3 (in Fig. 1, curves 1 through 4 correspond to H = −10, −3, 3, 10; in Fig. 2, curves 1 through 3 correspond to H = −∞, 0, ∞).

The class of flow under consideration is characterized by the existence of thresholds for motion and heat-exchange. For large thermal loads at the walls of the channel, in the case where the fluid is heated (large positive H), a piston-type motion occurs in which the fluid velocity in the center is constant and falls sharply to zero near the walls of the channel (see Fig. 1). The temperature profile for large H is given by $\theta(\xi) = 0.5\,(\xi^2 - 1)$ and

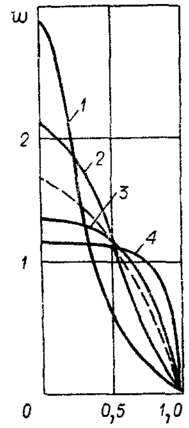

Fig. 1

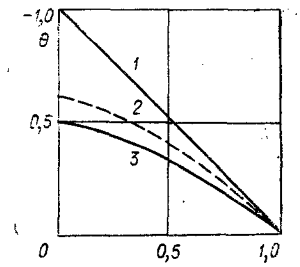

Fig. 2

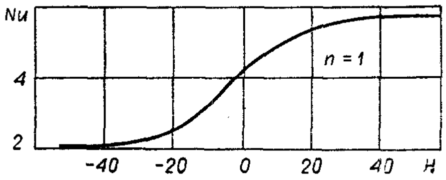

Fig. 3

the Nusselt number in this case approaches the limiting value of 6. Recall that for a Newtonian fluid with a constant viscosity, a parabolic velocity profile and a constant Nusselt number (Nu = 4.12) is characteristic [4].

For cooling of the fluid (H < 0) a boundary layer forms in the center such that near the walls of the channel the fluid is practically at rest for large values of H, and only near the axis does the velocity increase to large values. In this case the limiting temperature profile is given by $\theta(\xi) = \xi - 1$, $\mathrm{Nu} = 2$ (see Fig. 3). Thus for intermediate values of the constant H, the Nusselt number varies within the interval 2 < Nu < 6. In Figs. 1 and 2 the dashed lines show the classical solution for a Newtonian fluid with a constant viscosity.

We discuss some features of (4). Let $(H/n)\theta = v$, $\beta = -\frac{H}{n}\sqrt[n]{D}$. Then the boundary value problem (4) takes the form

$$v''' = \beta \sqrt[n]{\frac{\operatorname{sh}\delta\xi}{\operatorname{sh}\delta}}\, e^{v}, \quad v(1) = 0, \quad v'(0) = 0, \quad v''(1) = 0. \tag{8}$$

We construct the Green's function for the operator v''' = 0 with boundary conditions (8):

$$K(\xi, t) = \begin{cases} t - 0{,}5\,(t^2 + \xi^2), & \xi \leqslant t, \\ t - \xi t, & \xi \geqslant t \end{cases}$$

The solution of (8) is written as an integral equation

$$v(\xi) = \beta \int_0^1 K(\xi, t) \sqrt[n]{\frac{\operatorname{sh}\delta t}{\operatorname{sh}\delta}}\, e^{v(t)} dt. \tag{9}$$

We consider the case of cooling of the fluid β > 0 and the parameter β will be taken as given. Thus the drag coefficient of the channel or the pressure differential across its length is given.

Equation (8) corresponds to the following linear equation for the variation

$$\omega''' = \beta \sqrt[n]{\frac{\operatorname{sh}\delta\xi}{\operatorname{sh}\delta}}\, e^{v}\omega + \mu\omega', \quad \omega(1) = 0, \quad \omega'(0) = 0, \quad \omega''(1) = 0, \tag{10}$$

which describes the behavior of infinitesimal temperature pulses for the approximation where the velocity and pressure fields are quasistationary:

$$T(y, t) \approx \omega(y) e^{\mu\tau} (\tau = \lambda t/\rho c_p h^2).$$

If the spectrum of the boundary value problem (10) has all negative eigenvalues (μ < 0), viscous flow is stable with respect to thermal perturbations, and in the opposite case there is thermal instability.

Equations (8) and (10) differ from the classical equations describing thermal explosions [6] by the third order differential operator and the boundary conditions v''(1) =

$\omega''(1) = 0$. Because the Green's function of the iterated operator $L(v) = v'''$ is positive, the kernel of the integral equation (9) is oscillatory [8]. Therefore the results of the theory of thermal explosions (the existence of a critical β_* such that for $\beta < \beta_*$ the solution of (8) is positive and bounded, while such a solution does not exist for $\beta > \beta_*$, the nonuniqueness of the solution for $\beta < \beta_*$, the stability of the smallest positive solution, etc.) can be completely carried over to equations (8), (9), and (10).

For example, we show that β_* exists and estimate its upper bound. We assume that the solution of (8) exists for any positive value of β. Becasue $K(\xi, t) \geqslant 0$, we have $v(\xi, \beta) \geqslant 0$ for $\beta > 0$. We consider the following eigenvalue problem

$$z''' = -\mu \sqrt[n]{\frac{\operatorname{sh} \delta\xi}{\operatorname{sh} \delta}}\, z, \quad z(0) = 0, \quad z'(1) = 0, \quad z''(0) = 0. \tag{11}$$

Equation (11) is equivalent to a linear integral equation

$$z = \mu \int_0^1 G(\xi, t) \sqrt[n]{\frac{\operatorname{sh} \delta t}{\operatorname{sh} \delta}}\, z(t)\, dt, \tag{12}$$

where

$$G(\xi, t) = \begin{cases} \xi(1-t), & \xi \leqslant t, \\ \xi - 0.5(\xi^2 + t^2), & \xi \geqslant t. \end{cases}$$

Because the Green's function $G(\xi, t)$ of the iterated operator $L(z) = z'''$ is positive, the kernel of the integral equation (12) is oscillatory. According to [8], the eigenvalues of problem (12) satisfy the inequality $0 < \mu_0 < \mu_1 < \mu_2 \ldots$

The eigenfunction $z_0(\xi)$ corresponding to μ_0 does not have any zeros for $\xi \in (0, 1)$. Multiplying (8) by $z_0(\xi)$ and integrating using the boundary conditions (8), (11), we obtain the relation

$$\frac{\mu_0}{\beta} = \frac{\int_0^1 \sqrt[n]{\operatorname{sh} \delta\xi}\, z_0(\xi)\, e^{v(\xi)} d\xi}{\int_0^1 \sqrt[n]{\operatorname{sh} \delta\xi}\, z_0(\xi)\, v(\xi)\, d\xi}.$$

Because $v(\xi, \beta) \geqslant 0$ the inequality $e^v \geqslant ev$ is satisfied, from which it follows that $\beta \leqslant \mu_0 e^{-1}$. Hence for $\beta > \mu_0 e^{-1}$ a solution of boundary value problem (8) does not exist and therefore we obtain the estimate $\beta_* \leqslant \mu_0 e^{-1}$.

For small values of δ in the case of a Newtonian fluid ($n = 1$) we obtain the following values for the critical value of the pressure differential, the excess temperature, the heat flux, and Peclet number:

$$\beta_* = 3.55, \quad v_*(0) = 2.81, \quad H_* = 4, \quad \mathrm{Pe}_* = 4\delta^{-1}.$$

The dependence of β_* on the flow index n for large Pe ($\delta \to 0$) is shown in Fig. 4.

Thus in the nonisothermal flow of a viscous fluid in a channel of finite length whose walls are maintained at a temperature linearly dependent on the longitudinal coordinate, there exists a critical pressure differential above which the flow is not stationary. Physically this can be interpreted as a thermal explosion.

We discuss the mechanism of heat transfer. Heat enters the cooled section as a result of convection and is carried off toward the walls by molecular heat conduction; in addition, at the walls of the channel, there is heat exchange with the external medium such that a linear temperature distribution is established on the walls. For supercritical pressure differentials the heat given off by the fluid in the channel is not carried off to the walls because the transport of this heat by convection depends exponentially on the excess temperature, while the heat emitted into the external medium depends linearly on the excess temperature. A breakdown of the thermal stability of flow occurs, which is defined as a convective thermal explosion. It follows from the above analysis that the principal factor

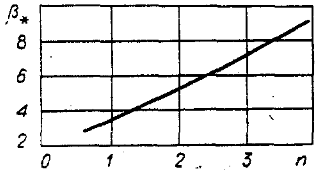

Fig. 4

in the formation of a thermal instability of viscous flow is the convective term in the equation of heat transport; this differs from a hydrodynamical thermal explosion where the principal factor is the dependence of the heat dissipation on temperature [5].

A similar situation arises in many physical problems involving propagation of heat (thermal explosions, thermal breakdown of dielectrics, etc.).

Our results can also be used in diffusion problems in the case where the viscosity of the fluid depends exponentially on the concentration of diffusing material.

Equations (8) and (9) written for the concentration of an impurity are exact (for the heat conduction problem (8) and (9) are approximate because heat dissipation is not taken into account).

We consider the motion in a plane channel of a viscoplastic medium of the Shvedov–Bingham type where the rheological equation has the form

$$\tau = -\tau_0 + \mu du/dy,\ \tau_0 \geqslant 0,\ |\tau| \geqslant \tau_0;\ \tau = -\tau_0,\ |\tau| \leqslant \tau_0.$$

We assume that in the temperature range under consideration $\tau_0 = \text{const}$, $\mu = \mu_0 e^{-\gamma(T-T_0)}$. As in the previous problem, the temperature is given by a linear distribution on the walls of the channel. In this case a structural motion of the viscoplastic medium occurs with viscous flow zones near the walls of the channel $(h^* \leqslant y \leqslant h)$ and an elastic zone $(0 \leqslant y \leqslant h^*)$. In the viscous flow region, by integrating the system of equations (1), we obtain the following expressions for the frictional stress and velocity gradient.

$$\tau = -\tau_0 + \frac{(\tau_w + \tau_0)\,\mathrm{sh}\,\gamma A(y - h^*)}{\mathrm{sh}\,\gamma A(h - h^*)},$$

$$\frac{du}{dy} = \frac{(\tau_w + \tau_0)}{\mu_w}\,\frac{\mathrm{sh}\,\gamma A(y - h^*)}{\mathrm{sh}\,\gamma A(h - h^*)}\,e^{\gamma T_1(y)},$$

where $T_1(y) = T(x, y) - T_w(x)$.

We obtain a boundary value problem for the excess temperature $v = \gamma T_1$:

$$v''' = B\frac{\mathrm{sh}\,\delta(\xi - a)}{\mathrm{sh}\,\delta(1 - a)}e^{v},\quad v(1) = 0,\quad v''(1) = 0,\quad v'(a) = av''(a), \tag{13}$$

where $\xi = y/h$; $a = h^*/h$; $B = \rho c_p A(\tau_w + \tau_0)\gamma h^3/\mu_w\lambda$; $\delta = \gamma Ah$; $h^* = \tau_0 L/\Delta p^*$; $\Delta p^*/L$ is the pressure differential on the axis of the elastic zone.

We introduce the Green's function $K_1(\xi, t)$ of operator $v''' = 0$ with boundary conditions (13). It can easily be shown that $K_1(\xi, t) \equiv K(\xi, t)$.

Hence we have the nonlinear integral equation

$$v(\xi) = B\int_a^1 K(\xi, t)\frac{\mathrm{sh}\,\delta(t - a)}{\mathrm{sh}\,\delta(1 - a)}e^{v(t)}dt. \tag{14}$$

We consider the case of cooling so that $B > 0$. Because the Green's functions for (9) and (14) are the same, there exists a B_* such that for $B > B_*$, a solution of the nonlinear integral equation (14) does not exist. Hence for $B > B_*$ we again have the situation where the thermal equilibrium between heat convection and molecular heat conduction cannot be maintained and this leads to a convection heat explosion. For subcritical flows ($B < B_*$) the solution of (14) can be constructed by series expansion in a small parameter. The following results are obtained for the velocity of the viscoplastic medium in the viscous flow zone:

$$w(\xi) = w_0(\xi) + Bw_1(\xi),\ w(\xi) = -u\mu_w/(\tau_w + \tau_0)h,$$
$$w_0(\xi) = -(1-a)^{-1}[0.5(\xi^2-1) - a(\xi-1)],$$
$$w_1(\xi) = -(12(1-a))^{-2}[\xi^6 - 6a\xi^5 + (6a^2+18a-9)\xi^4 - (8a^3 + $$
$$+ 24a^2 - 12)\xi^3 + (12a^4 + 12a^3 - 24a + 15)\xi^2 - (24a^4 - 48a^2 + $$
$$+ 30a)\xi + 12a^4 - 4a^3 - 30a^2 + 30a - 7].$$

Here $w_0(\xi)$ is the classical solution for the isothermal flow of a viscoplastic medium obtained by Volarovich and Gutkin [7].

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HEATING IN THE DEFORMATION OF A STRUCTURED FLOWING SYSTEM

L. M. Buchatskii, S. V. Maklakov,
A. M. Stolin, and S. I. Khudyaev

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The heating of a liquid by deformation may alter the flow curve (the relation between the stress and the shear rate). The relationship may become nonlinear even for a Newtonian liquid. This is related for example to the phenomenon of hydrodynamic thermal explosion [1, 2]. In the rheological processing of viscometric data, it is important to distinguish the heating effect from the effects of the internal properties of the liquid. For this purpose, either the experiment should be done under certainly isothermal conditions, which restricts the measurement range, or allowance for the heating must be made in the calculation of the characteristics. The latter is simplest to provide when there is spatial homogeneity in the temperature, which occurs for example in a constant-pressure (moment) viscometer [3].

Here we examine the behavior of a structured flowing system under conditions of heating and we distinguish the physically distinct flow states and determine the parameter ranges corresponding to the different types of rheological curve, and we also define critical conditions for structural ignition and extinction and for hydrodynamic thermal explosion in structured systems.

1. Formulation of the Problem. We consider the nonisothermal flow of a two-component liquid with mutual conversion of the components [4, 5]. The mathematical formulation includes not only the rheological and kinetic equations [4, 5] but also the heat-balance equation, which incorporates the dissipative heat production, the heat produced during the structural transformations, and the heat lost through the side walls:

$$\dot{\gamma} = \{aF_{01}\exp[\omega_1(T-T_0)] + (1-a)F_{02}\exp[\omega_2(T-T_0)]\}\tau; \quad (1.1)$$

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